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Weibull Stress/Strength Analysis with Non-Constant Shape Parameter

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Abstract: The paper presents the method to perform the Weibull stress-strength analysis when both the stress and the strength variables follow a Weibull distribution. And because the Weibull distribution does not have the additive property (known as Weibull closure property), then when both the stress and the strength variables present different shape parameter β, the Weibull stress-strength analysis is not defined. Therefore, based on the Weibull/Gumbel relationships and on the log-mean and log-standard deviation of the observed lifetime data, in this paper a common βc parameter is estimated and used to perform the corresponding Weibull stress/strength analysis. And since the derived βc parameter can be determined for any feasible pair of stress and strength β values, then the proposed method is always efficient to perform the corresponding stress/strength analysis. Finally, the method efficiency is shown through its application to a set of Weibull data also.

Keywords: Weibull distribution, Normal distribution, Stress-Strength analysis, Gumbel Distribution.

1. Introduction

It is a fact that, although the items are manufactured identically (same process, material, process, etc.), they have variation in their resistance (R_i) (Bickel et al., 2010). It is to say, the resistance of the units, say $R_1, R_2, \ldots, R_n$ is a continuous random variable (Levitin & Finkelstein, 2017). And as a consequence, their behavior must be represented by a probability distribution. Then, in reliability analysis, the fact that resistance (R_i) is a random variable is the first source of variation in the reliability analysis (Rinne, 2009). The second source of variation in reliability analysis occurs when the product is subjected to different levels of stress (E_i); i.e. $(E_1, E_2, \ldots, E_n)$ (Birolini, 2010). Therefore, because both the stress and the strength are random variables, then they have to be modeled through a probability distribution (Chiquet & Limnios, 2013). Even more, because the determination of the reliability of a product, element, system, etc., when they are subjected to a variant stress have to be performed by the stress-strength analysis (Babayi, Khorram, & Tondro, 2014), then the failure occurs when the stress exceeds the resistance $(P(s_i \geq S_i))$, or equivalently $R(t) = (P(s\_i \geq S\_i\ ))$ (Thoft-christensen, 2015).

On the other hand, because the stress-strength analysis models the random behavior of both the stress and the strength variables as probability density functions, where $(s_i\ y\ S_i)$ are both independent each other (Shodhganga, 2015), then the stress (s_i) is a variable that induces the failure of the product or element. In this way, the stress refers to a mechanical load, operating environment, temperature, electric current, etc., to which the product is subjected (Al-Mutairi, Ghitany, & Kundu, 2013). And the strngth (S_i) refers to the ability of a product, element, system, etc., to perform its design function satisfactorily when is subjected to external load and to the operating environment (NASA, 2007). Thus, a product is capable of performing its function if its resistance is greater than the applied stress (Kotz, Lumelskii, & Pensky, 2003). Therefore, the stress-strength analysis represents the probability that the stress is lower than the strength, or equivalently, the stress-strength analysis implies the algebraic sum of the stress and strength distributions (Babayi et al., 2014).

On the other hand, when the distribution of the stress and the distribution of the strength are both represented by a Weibull distribution, but they have different shape parameter, then because the closure property (Piña-Monarrez & Ortiz-Yañez, 2015), does not holds, then the stress/strength analysis is not defined. However, in this case, the common approach is to apply the normal stress-strength analysis. However, since there are not a close relationship between the normal parameters; mean μ and standard deviation σ with the Weibull parameters β and η, then the application of the normal stress-stress analysis when the stress and the strength variables follow a Weibull distribution is not efficient to determine the reliability of the designed component. Therefore, a method to perform the Weibull-Weibull strength-stress analysis by using the log-normal one is proposed. In this method, the Weibull and the log-normal parameters are related in such a way that a common Weibull β parameter is derived. The structure of the paper is as follows; Section 2 presents the stress-strength generalities. Section 3, presents the Weibull distribution generalizes and its closure property. In Section 4 an application of the actual stress-strength analysis is presented. In section 5 the common β parameter is determined, and the application is presented. Finally in section 6 the conclusions are given.

2. Stress-Strength Generalities

There are some applications were the product reliability depends on their physical inherent strength. Thus, if a stress level higher than the stress is applied then they break down. Therefore, if the random variable X represents the 'stress' and the random variable Y represents the 'strength' then the stress-strength reliability is denoted by the probability that Y>X $[R = P(Y < X)]$. In the stress-strength analysis the term stress is referred to the load that produces the failure and the strength is referred to the ability component to sustain the load.

On the other hand, in the stress-strength model the interference between the 'stress' variation and the 'strength' variation variables results in a statistical distribution. Thus, a natural scatter occurs in these variables when the two distributions interferes each other. And in particular, when the stress becomes higher than the strength, a failure occurs. In other words, when the probability density functions of both the stress and the strength are known, the component reliability may be analytically determined by its interference (see Fig.1). Seeing this let Y and X be two random variables such that Y represents "strength" and X represents "stress" and let Y, X follow a joint probability density function pdf $f(x, y)$. Then based on the $f(x, y)$, the reliability of the component is estimated as

$$R = P(X < Y) = \iint_{-\infty}^{x} f(y, x)dx\, dy \tag{3}$$

where $P(X < Y)$ is the probability that the strength exceeds the stress and $f(y, x)$ is the joint pdf of Y and X.

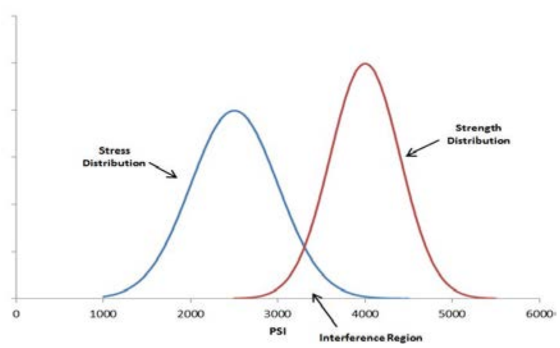

Figure 1. Stress and Strength Representation

The concept of stress-strength in engineering has been one of the deciding factors of failure of the devices. It has been customary to define safety factors for longer lives of the systems. This had been made in terms of the inherent components strength, and on the external stress being experienced by the systems. The safety factor is given by

$$SF = \frac{strength}{stress} \tag{4}$$

However, because in Eq.(2), the mean stress and mean strength values are used, then from figure 1, we can see that although the SF index defined in Eq.(2) is greater than one (SF>1), then because the scatter of both the stress and the strength, a region where both distributions overlap exists. This region is referred to as the "Interference Region", and it represents the failure probability. On the other hand, if the stress and the strength variables are both normally distributed, then the standard normal distribution and Z tables can be used to "quantitatively" determine the probability of failure by using the Z-statistic given by

$$Z = \frac{\mu_s - \mu_{sapp}}{\sqrt{\sigma_s^2 + \sigma_s^2{}_{sapp}}} \tag{5}$$

In Eq.(5) μS is the mean of the strength distribution, and "$\mu sapp$" is the mean of the stress distribution. Similarly, $\sigma S2$ is the variance of the strength distribution and $\sigma sapp2$ is the variance of the stress distribution (Quanterion Solutions Incorporated, 2014). And because Eq.(5) follows a normal distribution, then it possess the additive property; the sum of normal random variables is normal also. However, since the Weibull distribution does not possess this property, and there are not a relationship between the normal and Weibull parameters, Eq.(5) cannot be applied directly to perform the Weibull stress-strength analysis as it is shown in the next section.

3. Weibull Distribution Generalities

The Weibull distribution is widely used in reliability and life data analysis due to its versatility (McCool, 2012). Depending on the values of the parameters, the Weibull distribution can be used to model a variety of life behaviors (Piña-Monarrez, Ramos-Lopez, Alvarado-Iniesta, & Molina-Arredondo, 2016). An important aspect of the Weibull distribution is how the values of the shape parameter, β, and the scale parameter, η, affect the distribution characteristics (Cordeiro, Lima, Gomes, & Ortega, 2016). The Weibull shape parameter, β, is also known as the Weibull slope. This is because the value of β is equal to the slope of the line in a probability plot. Different values of the shape parameter can have high effects on the behavior of the distribution (Young, Lewis, Coleman, & Hunt, 2001). In fact, some values of the shape parameter will cause the distribution equations to reduce to those of other distributions. For example, when β = 1, the pdf of the three-parameter Weibull reduces to that of the two-parameter exponential distribution. The parameter β is a pure number (i.e., it is dimensionless). The following figure shows the effect of different values of the shape parameter, β, on the shape of the pdf (while keeping γ constant). One can see that the shape of the pdf can take a variety of forms based on the value of β (ReliaSoft Corporation, 1992).

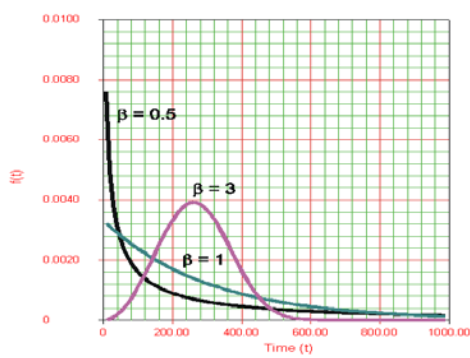

Figure 2. Effect of different values of the shape parameter, β

On the other hand, a change in the η value has the same effect on the distribution as a change on the abscissa scale. Increasing the value of η while holding β constant has the effect of stretching out the pdf (Pham, 2003). Thus, since the area under a pdf curve is a constant value of one, then the "peak" of the pdf curve will also decrease with an increase of η.

- If η is increased, while β and γ are kept constant, the distribution gets stretched out to the right and its height decreases, while maintaining its shape and location.
- If η is decreased, while β and γ are kept constant, the distribution gets pushed in towards the left (i.e., towards its beginning or towards 0 or γ), and its height increases.
- η has the same unit as T, such as hours, miles, cycles, actuations, etc.

The Weibull density function is given by:

$$f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} e^{-\left(\frac{t}{\eta}\right)^{\beta}} \tag{6}$$

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The cumulative Weibull distribution function is given by:

$$F(t) = 1 - e\left\{-(t/\eta)^{\beta}\right\} \tag{7}$$

And the Weibull reliability function is given by:

$$R(t) = e\left\{-(t/\eta)^{\beta}\right\} \tag{8}$$

Now let show that the Weibull distribution does not possess the additive property.

3.1 Weibull distribution closure property

For a system with n serial related components or having n independent failure modes, where each failure mode has an independent Weibull failure distribution with shape parameter β and scale parameter η then the joint failure rate function can be determined as (8):

$$P(\min(X_1, \ldots, X_n) > x) = P(X_1 > t, \ldots, x_n t) = \prod_{i=1}^{n} P(X_i > t) = \prod_{i=1}^{n} \exp\left[-\left(\frac{t}{\eta_i}\right)^{\beta}\right] =$$
$$\exp\ \left[-t^{\beta} \sum_{i=1}^{n} \frac{1}{\eta_i^{\beta}}\right] = \exp\left[-\left(\frac{t}{\eta*}\right)^{\beta}\ with\ \ \eta* = \left(\sum_{i=1}^{n} \frac{1}{\eta_i^{\beta}}\right)^{-1/\beta} \tag{8}$$

Since from Eq.(8), known as the reproductive Weibull property, we observe that the sum of Weibull random variables is defined only if they have the same β value. It is to say, if the failure modes are Weibull, but they have differing β parameter, then the system failure distribution will not be Weibull (Ebeling, 2010).

3.2 The General Weibull Stress-Strength Model

The use of Weibull distribution in reliability and quality control has been advocated by Kao (1959). The distribution is often suitable where the conditions of 'strict randomness' of the Exponential distribution are not satisfied. Considering the problem of finding the strength reliability of an item functioning until first failure, when both strength (Y) and stress (X) follow the Weibull distribution with the following Weibull pdf form:

$$f(x) = \beta\eta t^{\beta-1}\, e^{\eta t^{\beta}} \tag{9}$$

Hence, by considering some specific values of the shape parameters β_1 and β_2 as:

(i) Equal shape parameters ($\beta_1 = \beta_2$).

(ii) The shape parameter of strength is twice that of stress ($\beta_2 = 2\beta_1$).

(iii) The shape parameter of stress is twice that of strength ($\beta_1 = 2\beta_2$).

Then the Weibull stress-strength analysis for these cases can be performed. Here note that it is equivalent to say that the Weibull-Weibull stress-strength model is the same as that of the Exponential-Exponential stress-strength model, so long as the two Weibull distributions have the same shape parameter (Nadar & Kizilaslan, 2016). Now let present an application where the stress and the strength have different β values.

4. Weibull Stress-Strength Practical Case Using Weibull++

Here we will use the stress-strength analysis to estimate the reliability of a printer component. The stress is the distribution of the number of pages printed by users, and the strength is the distribution of the number of pages printed before the component failed during in-house testing. The warranty for the printer is one year, and the goal is to estimate the reliability of the component within this period, assess the confidence bounds on the reliability and to investigate the effect of the sample size on the confidence bounds (ReliaSoft Corporation, 2014). The numbers of pages printed per year from twenty different users are sorted and shown in Table 1. This information represents stress because it describes how much work the component has done in a given year. The numbers of pages printed before a failure occurred (for twenty printers) during in-house testing are sorted and shown in Table 2 (strength data).

Table 1. Numbers of pages printed per year

Stress: Number of Pages Printed		Strength: Number of Pages Printed to Failure	
17987	27274	27348	55948
19292	28352	39584	57868
19358	28434	39916	57904
20874	30172	43348	61944
22586	32456	46972	66712
22994	33038	47388	67476
23442	33856	47884	68712
24074	35692	48192	72584
25496	39162	49392	79924
25896	40642	49948	83084

From Table 1, the related stress and strength distributions using Weibull++ are shown in Figure 3 and Figure 4.

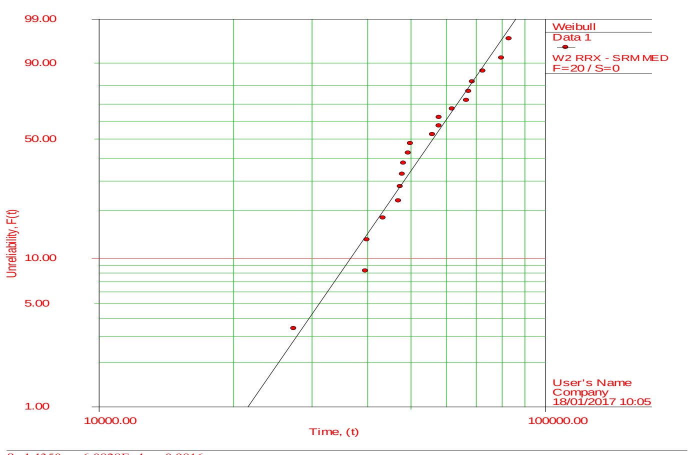

Figure 3. Stress Probability Weibull

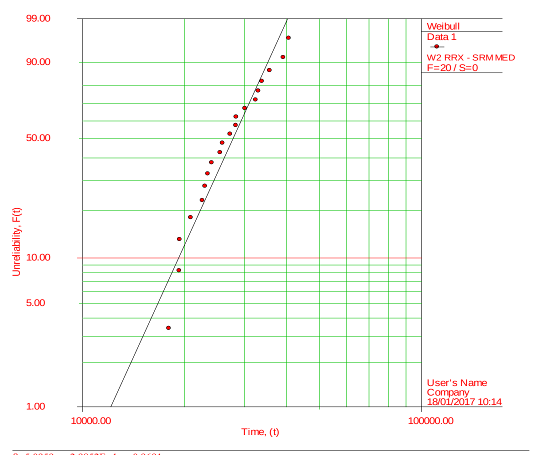

Figure 4. Strength Probability Weibull

5. Weibull Stress-Strength Proposed Reliability Estimation

The estimation of the reliability is based on the relations between the Weibull and Gumbel parameters given by (Crowder et al.,17 Section 2.4, pp 17).

$$\mu_{ev} = ln(\eta) \tag{12}$$

$$\sigma_{ev} = \frac{1}{\beta} \tag{13}$$

Thus, since by using the moment method (by NIST/SEMATECH18 Section 1.3.6.6.16), the Gumbel parameters defined in Eq.(10) and Eq.(11) in terms of the expected log-mean (μ_y) and log-standard deviation (σ_y) of the expected failure times are given by:

$$\mu_{ev} = E(x) = \mu_\gamma + \gamma\sigma_{ev} \tag{14}$$

$$\sigma_{ev} = \frac{\sqrt{6}}{\pi}\,\sigma_\gamma \tag{15}$$

Then from Equations (12) and (13) (the work of McCool,19 Section 6.7), an estimation of μ_y and σ_y is given by:

$$\widehat{\mu}_{\gamma} = \mu_{ev} - \gamma\sigma_{ev} \tag{16}$$

$$\hat{\sigma}_{\gamma} = \frac{\pi}{\sqrt{6}}\,\sigma_{ev} \tag{17}$$

In Equation (14), γ is the Euler's constant (γ = 0.577216…) (Piña-Monarrez, Ortiz-Yañez, & Rodríguez-Borbón, 2016). Therefore, based on Eq. (10, 12, 13, 16 and 17), to estimate a β common, by numeric solution, stress-strength reliability can also be estimated by formulating a new Weibull distribution as it is shown in Table 2.

From Table 3, it is observed that the estimated reliability R(t) using the Weibull ++ software and the results obtained using the proposed method to estimate a common shape parameter β, is the same R(t). Thus, we conclude that by using a common *βc* parameter, we can determine efficiently the stress-strength Weibull reliability when the stress and strength presents different *βs* parameters.

Table 2. Reliability comparison between methods with η constant:

Standard Method					**Proposed Method**				
β_S	η_s	β_s	η_s	$R(t)$	β_c	η_s	β_c	η_s	$R(t)$
1	60828	1.5	29852	66.88	0.9873	60828	0.9873	29852	66.88
2	60828	2.5	29852	81.11	2.0472	60828	2.0472	29852	81.11
3	60828	3.5	29852	89.79	3.0544	60828	3.0544	29852	89.79
4.43	60828	5.08	29852	96.08	4.4944	60828	4.4944	29852	96.08
5	60828	6	29852	97.38	5.0793	60828	5.0793	29852	97.38

6. Conclusions

By using Eq. (12), (13), (16) and (17), the initial β value to estimate numerically from Eq. (10) a common βc value is used. Due to the βc value represents the stress and strength β values, then it is efficient to calculated the stress-strength reliability $R(t)$, when both the stress and the strength variables present different β parameters. And this form the βc value is helpful in the decision-making process. It is important to remember that because the closure property, then in the Weibull stress-strength analysis, the standard stress-strength methodology has a close form, only when the stress and strength parameters have the same β value, or the half of the β value of the stress is equal to the β value of the strength, and vice versa. As a result, the use of a βc value is useful for the reliability practitioners.

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